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DIFFERENTIAL QUADRATURE METHOD FOR THICK SYMMETRIC CROSS-PLY LAMINATES WITH FIRST-ORDER SHEAR FLEXIBILITY

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Abstract—A global numerical technique, the differential quadrature (DQ) method, is examined here for its suitability to solve the boundary-value problem of symmetric cross-ply laminates using the first-order shear deformation plate theory by Whitney and Pagano [*J. Appl. Mech.* **37,** 1031–1036 (1970)]. The bending behaviours of symmetric cross-ply laminates, subject to different boundary constraints, are investigated. In this study, the method is used to transform the sets of governing differential equations and boundary conditions of the laminated plates into sets of linear algebraic equations. Boundary conditions along the edges are implemented through the discrete grid points by constraining the displacements, bending moments and rotations. The theoretical formulations and solution procedures of the method are illustrated through solving several numerical examples. The accuracy and validity of the present formulation, if available, are examined by direct comparison with the known values. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

The unique properties of advanced composite materials have resulted in extensive applications of laminated plates to aerospace, automobile, shipbuilding and nuclear industries. It is well known that the classical laminated plate theory, based on the Kirchhoff hypothesis, yields only acceptable results for thin composite laminates (Liew, 1992). Therefore considerable efforts have been made to improve prediction ability of the transverse shear-strain distribution in laminated plate theory since shear deformation plays a significant role in the prediction of shear stress for laminated plates. Numerous researchers, for example, Whitney and Pagano (1970), Bert and Chen (1978), and Reddy and Chao (1981), have presented closed-form solutions for laminated plates with at least a pair of opposite edges simply supported using first-order shear deformation theory. Various improved higher-order shear deformation theories are also proposed for the analysis of laminated plates (Reddy, 1984; Pandya and Kant, 1988; Liu *et al.*, 1994). Three-dimensional solutions to laminated plates are also available, for example, Pagano (1969, 1970), Srinivas and Rao (1970), Pagano and Hatfield (1972), Noor and Burton (1990), Tungikar and Rao (1994), Wang and Tarn (1994), as a benchmark check for the verification of the shear deformation theories.

Although much work has been done on the modelling of laminated composite plates, closed-form solutions for laminates using the first-order shear deformation theory (Whitney and Pagano, 1970), higher-order shear deformation theory (Reddy, 1984) and three-dimensional elasticity theory (Srinivas and Rao, 1970) are only restricted to plates having at least a pair of opposite edges simply supported. For other combinations of boundary conditions, only approximate solutions could be obtained using some numerical methods. Due to the advancement of computer technology, some numerical methods, such as the finite element method, boundary element method and finite difference method, have been widely used in simulating the response of structures. These numerical methods have reported useful information in the modelling of laminated plate responses.

As an alternative to these conventional numerical methods, the differential quadrature procedure has been proposed for solving engineering problems (Bellman and Casti, 1971; Bellman *et al.*, 1972; Civan and Sliepcevich, 1984; Quan and Chang, 1989; Shu and

Richards, 1992). In this paper, we attempt to implement this method to solve static responses of thick symmetric cross-ply laminated plates subjected to transverse loads. The plates are described by the first-order shear deformation theory (Whitney and Pagano, 1970) and with arbitrary boundary conditions. The DQ method transforms the system equations, governing differential equations and boundary conditions, into a set of linear algebraic equations in terms of unknown function values at discrete points of the solution domain. The response quantities of the laminates are obtained by solving these algebraic equations. Effects of various factors, such as relative thickness, boundary conditions, and aspect ratios, on the static responses of the laminates are investigated. To evaluate the accuracy of this numerical method to thick laminated plates, the results are compared with some closed-form and numerical solutions, if they are available.

2. DIFFERENTIAL QUADRATURE METHOD

The differential quadrature procedure was first proposed as a simple and efficient numerical approach to solve linear and non-linear partial differential equations of initial value problems (Bellman and Casti, 1971; Bellman *et al.*, 1972; Bellman, 1973). It was further extended to multidimensional problems by Civan and Sliepcevich (1984), to structural problems by Bert *et al.* (1988) and Sherbourne and Pandey (1991), and to thin orthotropic and laminated plates by Bert *et al.* (1989) and Farsa *et al.* (1993).

The basic idea of differential quadrature method (DQ method) rests on the approximation of partial derivatives of a function with respect to a variable at any discrete point as the weighted linear sums of the function values at all the discrete points chosen in the overall domain of that variable. According to this basic idea, we can solve a two-dimensional problem with basic variables in a rectangular domain. Supposing that there are N_x grid points in the x-direction and N_y in the y-direction with $x_1, x_2, \ldots, x_{N_x}$ and $y_1, y_2, \ldots, y_{N_y}$ as the coordinates, the nth-order partial derivative of $f(x, y)$ with respect to x, the mth-order partial derivative of $f(x, y)$ with respect to y and the $(n+m)$th-order partial derivative of $f(x, y)$ with respect to both x and y can be expressed discretely at the point (x_i, y_j) as:

$$f_x^{(n)}(x_i, y_j) = \sum_{k=1}^{N_x} C_{ik}^{(n)} f(x_k, y_j); \quad n = 1, 2, \ldots, N_x - 1 \tag{1a}$$

$$f_y^{(m)}(x_i, y_j) = \sum_{k=1}^{N_y} \bar{C}_{jk}^{(m)} f(x_i, y_k); \quad m = 1, 2, \ldots, N_y - 1 \tag{1b}$$

$$f_{xy}^{(n+m)}(x_i, y_j) = \sum_{k=1}^{N_x} C_{ik}^{(n)} \sum_{l=1}^{N_y} \bar{C}_{jl}^{(m)} f(x_k, y_l) \tag{1c}$$

for $i = 1, 2, \ldots, N_x$, and $j = 1, 2, \ldots, N_y$

where $C_{ij}^{(n)}$ and $\bar{C}_{ij}^{(m)}$ are weighting coefficients associated with nth-order partial derivative of $f(x, y)$ with respect to x at the discrete point x_i and mth-order derivative with respect to y at y_i, respectively.

In the original DQ method, there are two approaches to determine the weighting coefficients for the first order of derivatives (Bellman *et al.*, 1972). The first approach, which has been widely adopted (Bellman, 1973; Civan and Sliepcevich, 1984; Bert *et al.*, 1988, 1989; Sherbourne and Pandey, 1991; Farsa *et al.*, 1993), requires that the eqn (1a) or (1b) be exact for all power polynomials of degree less than or equal to $(N_x - 1)$ or $(N_y - 1)$, respectively (Bellman *et al.*, 1972; Civan and Sliepcevich, 1984). This results in a set of linear algebraic equations which are solved to obtain the weighting coefficients. The second approach is to choose the roots of shifted Legendre polynomials as the coordinates of the grid points, which restricts the flexibility of choice of grid points such that as the number of grid points is set, the distributions of grid points are exactly the same for various problems (Bellman *et al.*, 1972). Following the same fundamental concept, Quan and Chang (1989) established a set of simple algebraic expressions to calculate the weighting coefficients for the first- and second-order derivatives, followed later by Shu and Richards (1990, 1992), who derived a simple algebraic expression and a recurrence formula to calculate the higher-order weighting coefficients. It is noted that for the first- and second-order derivatives, the weighting coefficients derived by Shu and Richards (1990, 1992) are identical to those given earlier by Quan and Chang (1989). In this paper, we employ the generalised and simplified version of the DQ method (Quan and Chang, 1989; Shu and Richards, 1990, 1992) throughout the present investigation.

Using the generalised and simplified DQ method, the weighting coefficients in eqn (1), i.e. $C_{ij}^{(n)}$ and $\bar{C}_{ij}^{(m)}$, can be determined as follows:

$$C_{ij}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)}; \quad i,j = 1,2,\ldots,N_x, \quad \text{but } j \neq i, \tag{2a}$$

$$\bar{C}_{ij}^{(1)} = \frac{P^{(1)}(y_i)}{(y_i - y_j)P^{(1)}(y_j)}; \quad i,j = 1,2,\ldots,N_y, \quad \text{but } j \neq i, \tag{2b}$$

where

$$M^{(1)}(x_i) = \prod_{j=1,j\neq i}^{N_x} (x_i - x_j), \tag{3a}$$

$$P^{(1)}(y_i) = \prod_{j=1,j\neq i}^{N_y} (y_i - y_j), \tag{3b}$$

and

$$C_{ij}^{(n)} = n\left(C_{ii}^{(n-1)}C_{ij}^{(1)} - \frac{C_{ij}^{(n-1)}}{x_i - x_j}\right) \tag{4a}$$

for $i,j = 1,2,\ldots,N_x$, but $j \neq i$; and $n = 2,3,\ldots,N_y - 1$

$$\bar{C}_{ij}^{(m)} = m\left(\bar{C}_{ii}^{(m-1)}\bar{C}_{ij}^{(1)} - \frac{\bar{C}_{ij}^{(m-1)}}{y_i - y_j}\right) \tag{4b}$$

for $i,j = 1,2,\ldots,N_y$, but $j \neq i$; and $m = 2,3,\ldots,N_y - 1$

$$C_{ii}^{(n)} = -\sum_{j=1,j\neq i}^{N_x} C_{ij}^{(n)}; \quad i = 1,2,\ldots,N_x, \text{ and } n = 1,2,\ldots,N_x - 1 \tag{5a}$$

$$\bar{C}_{ii}^{(m)} = -\sum_{j=1,j\neq i}^{N_y} \bar{C}_{ij}^{(m)}; \quad i = 1,2,\ldots,N_y, \text{ and } m = 1,2,\ldots,N_y - 1. \tag{5b}$$

Once the functional values at all grid points are calculated, the value at any point could be readily obtained in terms of the polynomial approximation:

$$f(x, y_j) = \sum_{i=1}^{N_x} f(x_i, y_j) r_i(x) \tag{6a}$$

$$f(x_i, y) = \sum_{j=1}^{N_y} f(x_i, y_j) s_j(y) \tag{6b}$$

$$f(x, y) = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} f(x_i, y_j) r_i(x) s_j(y) \tag{6c}$$

where $r_i(x)$ and $s_j(y)$ are the Lagrange interpolation polynomials along the x- and y-directions, respectively, and given in the following forms:

$$r_i(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_{N_x})}{M^{(1)}(x_i)} \tag{7a}$$

$$s_j(y) = \frac{(y-y_1)(y-y_2)\dots(y-y_{j-1})(y-y_{j+1})\dots(y-y_{N_y})}{P^{(1)}(y_j)}. \tag{7b}$$

In a similar manner, derivatives of $f(x, y)$ with respect to x or y at any point, which does not belong to any grid point, could be easily obtained in terms of the respective derivative values at all grid points.

It should be noted that once the grid of a domain to be analysed is set, the weighting coefficients are determined using eqns (2)–(5) regardless of how many unknown functions exist in a set of partial differential equations.

3. MATHEMATICAL FORMULATION

3.1. *Governing differential equations*

Consider a flat, moderately thick, symmetric cross-ply rectangular laminate with the midplane as the x–y coordinate plane. It is assumed in this analysis that the layers are perfectly bonded, with each layer being of uniform thickness. In terms of the first-order shear deformation theory (Whitney and Pagano, 1970), the governing differential equations for a symmetric cross-ply and specially orthotropic laminated plate subjected to a lateral load are:

$$D_{11}\frac{\partial^2\psi_x}{\partial x^2} + D_{66}\frac{\partial^2\psi_x}{\partial y^2} + (D_{12}+D_{66})\frac{\partial^2\psi_y}{\partial x\,\partial y} - A_{55}\left(\psi_x + \frac{\partial w}{\partial x}\right) = 0, \tag{8a}$$

$$(D_{12}+D_{66})\frac{\partial^2\psi_x}{\partial x\,\partial y} + D_{66}\frac{\partial^2\psi_y}{\partial x^2} + D_{22}\frac{\partial^2\psi_y}{\partial y^2} - A_{44}\left(\psi_y + \frac{\partial w}{\partial y}\right) = 0, \tag{8b}$$

$$A_{55}\left(\frac{\partial\psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + A_{44}\left(\frac{\partial\psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) + p(x, y) = 0, \tag{8c}$$

where $w(x, y)$ is the transverse deflection; $\psi_x(x, y)$ and $\psi_y(x, y)$ are the rotations of the normal about x- and y-axes; and $p(x, y)$ is the surface load intensity. The laminate stiffness coefficients, A_{44}, A_{55}, D_{11}, D_{22}, D_{66} and D_{12}, are clearly defined in Vinson and Sierakowski (1986).

In terms of force resultants and deformation variables, the following formulae are established:

$$M_x = D_{11}\frac{\partial\psi_x}{\partial x} + D_{12}\frac{\partial\psi_y}{\partial y}, \tag{9a}$$

$$M_y = D_{12}\frac{\partial\psi_x}{\partial x} + D_{22}\frac{\partial\psi_y}{\partial y}, \tag{9b}$$

$$M_{xy} = D_{66}\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\right), \tag{9c}$$

$$Q_x = A_{55}\left(\frac{\partial w}{\partial x} + \psi_x\right), \tag{9d}$$

$$Q_y = A_{44}\left(\frac{\partial w}{\partial y} + \psi_y\right), \tag{9e}$$

where M_x, M_y and M_{xy} are the moment resultants; and Q_x and Q_y are the shear resultants.

3.2. *Normalisation of governing equations*

To normalise the governing equations, the following nondimensional parameters are adopted:

$$X = x/a; \quad Y = y/b; \quad W = w/h; \quad \Psi_X = \psi_x; \quad \Psi_Y = \psi_y; \tag{10a}$$

$$\beta = a/b; \quad \gamma = h/b; \quad \delta = h/a; \tag{10b}$$

$$\lambda_{22} = D_{22}/D_{11}; \quad \lambda_{12} = D_{12}/D_{11}; \quad \lambda_{66} = D_{66}/D_{11}; \quad \lambda_{55} = A_{55}a^2/D_{11};$$
$$\lambda_{44} = A_{44}a^2/D_{11}; \tag{10c}$$

where a and b are the length and the width of the plate.

Using eqn (10), the governing differential equations (8) become:

$$\frac{\partial^2\Psi_X}{\partial X^2} + \lambda_{66}\beta^2\frac{\partial^2\Psi_X}{\partial Y^2} - \lambda_{55}\Psi_X + (\lambda_{12}+\lambda_{66})\beta\frac{\partial^2\Psi_Y}{\partial X\partial Y} - \lambda_{55}\delta\frac{\partial W}{\partial X} = 0, \tag{11a}$$

$$(\lambda_{12}+\lambda_{66})\beta\frac{\partial^2\Psi_X}{\partial X\partial Y} + \lambda_{22}\beta^2\frac{\partial^2\Psi_Y}{\partial Y^2} + \lambda_{66}\frac{\partial^2\Psi_Y}{\partial X^2} - \lambda_{44}\Psi_Y - \lambda_{44}\gamma\frac{\partial W}{\partial Y} = 0, \tag{11b}$$

$$\lambda_{55}\delta\frac{\partial^2 W}{\partial X^2} + \lambda_{44}\gamma\beta\frac{\partial^2 W}{\partial Y^2} + \lambda_{55}\frac{\partial\Psi_X}{\partial X} + \lambda_{44}\beta\frac{\partial\Psi_Y}{\partial Y} + \frac{a^3 p}{D_{11}} = 0. \tag{11c}$$

3.3. *Discretisation of normalised governing equations*

According to the DQ method, the normalised governing differential equations (11) take the following discrete forms:

$$\sum_{k=1}^{N_x} C_{ik}^{(2)}(\Psi_X)_{kj} + \lambda_{66}\beta^2\sum_{m=1}^{N_y}\bar{C}_{jm}^{(2)}(\Psi_X)_{im} - \lambda_{55}(\Psi_X)_{ij}$$
$$+ (\lambda_{12}+\lambda_{66})\beta\sum_{k=1}^{N_x}C_{ik}^{(1)}\sum_{m=1}^{N_y}\bar{C}_{jm}^{(1)}(\Psi_Y)_{km} - \lambda_{55}\delta\sum_{k=1}^{N_x}C_{ik}^{(1)}W_{kj} = 0 \tag{12a}$$

$$(\lambda_{12}+\lambda_{66})\beta\sum_{k=1}^{N_x}C_{ik}^{(1)}\sum_{m=1}^{N_y}\bar{C}_{jm}^{(1)}(\Psi_X)_{km} + \lambda_{22}\beta^2\sum_{m=1}^{N_y}\bar{C}_{jm}^{(2)}(\Psi_Y)_{im}$$
$$+ \lambda_{66}\sum_{k=1}^{N_x}C_{ik}^{(2)}(\Psi_Y)_{kj} - \lambda_{44}(\Psi_Y)_{ij} - \lambda_{44}\gamma\sum_{m=1}^{N_y}\bar{C}_{jm}^{(1)}W_{im} = 0 \tag{12b}$$

$$\lambda_{55}\delta\sum_{k=1}^{N_x}C_{ik}^{(2)}W_{kj} + \lambda_{44}\gamma\beta\sum_{m=1}^{N_y}\bar{C}_{jm}^{(2)}W_{im} + \lambda_{55}\sum_{k=1}^{N_x}C_{ik}^{(1)}(\Psi_X)_{kj}$$
$$+ \lambda_{44}\beta\sum_{m=1}^{N_y}\bar{C}_{jm}^{(1)}(\Psi_Y)_{im} + \frac{a^3}{D_{11}}p_{ij} = 0 \tag{12c}$$

where $i = 1, 2, \ldots, N_x$ and $j = 1, 2, \ldots, N_y$. $C_{rs}^{(n)}$ and $\bar{C}_{rs}^{(n)}$, which can be determined according to eqns (2)–(5), are the weighting coefficients for the nth-order partial derivatives of W, Ψ_X and Ψ_Y with respect to X and Y, respectively.

The stress–displacement relationship (9) can also be normalised in terms of eqn (10), then discretised as:

$$(\bar{M}_X)_{ij} = \sum_{k=1}^{N_x} C_{ik}^{(1)}(\Psi_X)_{kj} + \lambda_{12}\beta \sum_{m=1}^{N_y} \bar{C}_{jm}^{(1)}(\Psi_Y)_{im} \quad (13a)$$

$$(\bar{M}_Y)_{ij} = \lambda_{12} \sum_{k=1}^{N_x} C_{ik}^{(1)}(\Psi_X)_{kj} + \lambda_{22}\beta \sum_{m=1}^{N_y} \bar{C}_{jm}^{(1)}(\Psi_Y)_{im} \quad (13b)$$

$$(\bar{M}_{XY})_{ij} = \lambda_{66} \sum_{k=1}^{N_x} C_{ik}^{(1)}(\Psi_Y)_{kj} + \lambda_{66}\beta \sum_{m=1}^{N_y} \bar{C}_{jm}^{(1)}(\Psi_X)_{im} \quad (13c)$$

$$(\bar{Q}_X)_{ij} = \lambda_{55}(\Psi_X)_{ij} + \lambda_{55}\delta \sum_{m=1}^{N_x} C_{ik}^{(1)} W_{kj} \quad (13d)$$

$$(\bar{Q}_Y)_{ij} = \lambda_{44}(\Psi_Y)_{ij} + \lambda_{44}\gamma \sum_{m=1}^{N_y} \bar{C}_{jm}^{(1)} W_{im} \quad (13e)$$

where

$$\bar{M}_X = \frac{M_x a}{D_{11}}; \quad \bar{M}_Y = \frac{M_y a}{D_{11}}; \quad \bar{M}_{XY} = \frac{M_{xy} a}{D_{11}}; \quad \bar{Q}_X = \frac{Q_x a^2}{D_{11}}; \quad \bar{Q}_Y = \frac{Q_y a^2}{D_{11}}. \quad (14)$$

3.4. *Normalisation of boundary conditions*

Three types of boundary conditions are considered herein. For example, for an edge with x = constant, they are:

(a) *simply supported edge* (S): $w = M_x = \psi_y = 0,$ (15)

(b) *clamped edge* (C): $w = \psi_x = \psi_y = 0,$ (16)

(c) *free edge* (F): $Q_x = M_x = M_{xy} = 0.$ (17)

The boundary conditions are normalised according to eqn (10) and expressed by the basic variables W, Ψ_X and Ψ_Y. Thus the boundary conditions in eqns (15)–(17) for an edge X = constant are written as:

(a) *simply supported edge* (S):

$$W = 0; \quad \frac{\partial \Psi_X}{\partial X} + \lambda_{12}\beta\frac{\partial \Psi_Y}{\partial Y} = 0; \quad \Psi_Y = 0; \quad (18)$$

(c) *clamped edge* (C):

$$W = 0; \quad \Psi_X = 0; \quad \Psi_Y = 0; \quad (19)$$

(d) *free edge* (F):

$$\delta\frac{\partial W}{\partial X} + \Psi_X = 0; \quad \frac{\partial \Psi_X}{\partial X} + \lambda_{12}\beta\frac{\partial \Psi_Y}{\partial Y} = 0; \quad \beta\frac{\partial \Psi_X}{\partial Y} + \frac{\partial \Psi_Y}{\partial X} = 0. \quad (20)$$

3.5. *Discretisation of normalised boundary conditions*

The various boundary conditions, represented mathematically by eqns (18)–(20) for an edge of X = const., can be expressed into the following discrete forms. For example, at the edge $X = 0$:

• (S)

$$W_{1j} = 0 \quad \text{for } j = 1,2,\ldots,N_y, \tag{21a}$$

$$\sum_{k=1}^{N_x} C_{1k}^{(1)}(\Psi_X)_{kj} + \lambda_{12}\beta \sum_{m=1}^{N_y} \bar{C}_{jm}^{(1)}(\Psi_Y)_{1m} = 0 \quad \text{for } j = 1,2,\ldots,N_y, \tag{21b}$$

$$(\Psi_Y)_{1j} = 0 \quad \text{for } j = 1,2,\ldots,N_y, \tag{21c}$$

• (C)

$$W_{1j} = 0 \quad \text{for } j = 1,2,\ldots,N_y, \tag{22a}$$

$$(\Psi_X)_{1j} = 0 \quad \text{for } j = 1,2,\ldots,N_y, \tag{22b}$$

$$(\Psi_Y)_{1j} = 0 \quad \text{for } j = 1,2,\ldots,N_y, \tag{22c}$$

• (F)

$$\delta \sum_{k=1}^{N_x} C_{1k}^{(1)} W_{kj} + (\Psi_X)_{1j} = 0 \quad \text{for } j = 1,2,\ldots,N_y, \tag{23a}$$

$$\sum_{k=1}^{N_x} C_{1k}^{(1)}(\Psi_X)_{kj} + \lambda_{12}\beta \sum_{m=1}^{N_y} \bar{C}_{jm}^{(1)}(\Psi_Y)_{1m} = 0 \quad \text{for } j = 1,2,\ldots,N_y, \tag{23b}$$

$$\beta \sum_{m=1}^{N_y} \bar{C}_{jm}^{(1)}(\Psi_X)_{1m} + \sum_{k=1}^{N_x} C_{1k}^{(1)}(\Psi_Y)_{kj} = 0 \quad \text{for } j = 1,2,\ldots,N_y. \tag{23c}$$

The discretised boundary conditions for the edge $Y = 0$ are:

• (S)

$$W_{i1} = 0 \quad \text{for } i = 1,2,\ldots,N_x, \tag{24a}$$

$$(\Psi_X)_{i1} = 0 \quad \text{for } i = 1,2,\ldots,N_x, \tag{24b}$$

$$\lambda_{12} \sum_{k=1}^{N_x} C_{1i}^{(1)}(\Psi_X)_{k1} + \lambda_{22}\beta \sum_{m=1}^{N_y} \bar{C}_{1m}^{(1)}(\Psi_Y)_{im} = 0 \quad \text{for } i = 1,2,\ldots,N_x, \tag{24c}$$

• (C)

$$W_{i1} = 0 \quad \text{for } i = 1,2,\ldots,N_x, \tag{25a}$$

$$(\Psi_X)_{i1} = 0 \quad \text{for } i = 1,2,\ldots,N_x, \tag{25b}$$

$$(\Psi_Y)_{i1} = 0 \quad \text{for } i = 1,2,\ldots,N_x, \tag{25c}$$

• (F)

$$\gamma \sum_{k=1}^{N_y} \bar{C}_{1m}^{(1)} W_{im} + (\Psi_Y)_{i1} = 0 \quad \text{for } i = 1,2,\ldots,N_x, \tag{26a}$$

$$\beta \sum_{m=1}^{N_y} \bar{C}_{1m}^{(1)}(\Psi_X)_{im} + \sum_{k=1}^{N_x} C_{ik}^{(1)}(\Psi_Y)_{k1} = 0 \quad \text{for } i = 1,2,\ldots,N_x, \tag{26b}$$

$$\lambda_{22}\beta \sum_{m=1}^{N_y} \bar{C}_{1m}^{(1)}(\Psi_Y)_{im} + \lambda_{12} \sum_{k=1}^{N_x} C_{ik}^{(1)}(\Psi_X)_{k1} = 0 \quad \text{for } i = 1,2,\ldots,N_x. \tag{26c}$$

Similarly, for the edge at $X = 1$, the discrete boundary conditions can be obtained according to eqns (21)–(23) except that all the subscripts of 1 in eqns (21)–(23) are substituted by N_x (but kept $j = 1,2,\ldots,N_y$). The boundary conditions and their corresponding normalized and discrete forms at the edge $Y = 1$ could be readily written in the same manner.

4. RESULTS AND DISCUSSION

The DQ method described in the previous sections is written into a code "DQ" and run in a personal computer to solve the laminated plate problems. For convenience, only uniformly distributed grid points are adopted. The notation, for instance, SCFC denotes a plate with edges $X = 0$, $Y = 0$, $X = 1$ and $Y = 1$ having simply supported, clamped, free and clamped boundary conditions, respectively. This boundary convention is used throughout the present study. For simplicity, the shear correction factors κ_4^2 and κ_5^2 are assumed to be 5/6, and the transverse stresses (τ_{xz}, τ_{yz}) are calculated through the plate constitutive equations.

Two types of loading conditions are considered:

(a) *uniform pressure* (UN): $p(x,y) = p_0$

(b) *sinusoidal transverse load* (SS): $p(x,y) = p_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$.

The following sets of material properties are used in the calculations:

(a) Material I (Reddy, 1984):

$$\frac{E_1}{E_2} = 2, \quad \frac{G_{12}}{E_2} = \frac{G_{13}}{E_2} = 0.5, \quad \frac{G_{23}}{E_2} = 0.2,$$

$$\nu_{12} = 0.25, \quad E_2 = 7\,\text{GPa};$$

(b) Material II (Liu *et al.*, 1994):

$$E_1 = 20.83\times10^6\,\text{psi}, \quad E_2 = 10.94\times10^6\,\text{psi}, \quad G_{12} = 6.10\times10^6\,\text{psi},$$

$$G_{13} = 3.71\times10^6\,\text{psi}, \quad G_{23} = 6.19\times10^6\,\text{psi}, \quad \nu_{12} = 0.44;$$

where E_1 and E_2 are Young's moduli along and transverse to the fibre, respectively; G_{12} is in-plane shear modulus, while G_{23} and G_{13} are transverse shear moduli; ν_{12} and ν_{21} are Poisson's ratios along and transverse to the fibre. The subscripts 1, 2 and 3 are the principal directions of material, which are along and transverse to the fibre orientation in the plate plane, and perpendicular to the plate, respectively.

Unless specified otherwise, the locations, i.e. x, y and z-coordinates, for all deflections and stresses for the present evaluations are as follows:

- Transverse displacement (w): (0.5a, 0.5b);
- In-plane normal stress (σ_{xx}): (0.5a, 0.5b, 0.5h);
- In-plane normal stress (σ_{yy}): (0.5a, 0.5b, 0.25h);
- In-plane shear stress (τ_{xy}): (0.0a, 0.0b, 0.5h);
- Transverse shear stress (τ_{xz}): (0.0a, 0.5b, 0.0h);
- Transverse shear stress (τ_{yz}): (0.5a, 0.0b, 0.0h);

Several examples are solved to demonstrate the accuracy and convergent properties of the solutions obtained using the DQ method.

Table 1 presents the numerical results using different grid points for a square simply supported four-layer cross-ply (0°/90°/90°/0°) laminated plate subject to sinusoidal transverse load. The grid points are varying from a range of 5×5 to 13×13. It is observed that the deflection and stress components converge by using 11×11 grid points. The variation of thickness-to-span ratio ($h/a = 0.01$–0.25) only has a little effect on the rate of convergence.

The results, in Table 1, are compared respectively to the closed-form solutions by Reddy and Chao (1981) using the first-order shear deformation theory, the finite element Mindlin solutions by Pandya and Kant (1988), and three-dimensional elasticity solutions by Pagano and Hatfield (1972). Compared with the data given by Reddy and Chao (1981), it is shown that the present result, except the stress components τ_{xz} and τ_{yz}, are in close agreement. The discrepancy between the DQ method solutions and those by Reddy and Chao (1981) for the shear stresses τ_{xz} and τ_{yz} results from the different theories used to obtain the stresses. The values of Reddy and Chao are obtained using the three-dimensional elasticity equilibrium equations, while the present results are obtained using the plate constitutive equations. The present results are compared with those given by Pandya and Kant (1988) who calculated the shear stresses τ_{xz} and τ_{yz} from the plate constitutive

Table 1. Convergence and accuracy—deflections and stresses in a simply supported four-layer cross-ply (0°/90°/90°/0°) square laminate under sinusoidal transverse load ($h_i = h/4$, $i = 1, \ldots, 4$; Material I)

h/a	Grid points	$w \times m_1$†	$\sigma_{xx} \times m_2$†	$\sigma_{yy} \times m_2$	$\tau_{yz} \times m_3$†	$\tau_{zx} \times m_3$	$\tau_{xy} \times m_2$
0.25	5×5	1.7653	0.4534	0.6107	0.1513	0.1129	−0.0335
	7×7	1.7064	0.4034	0.5745	0.1640	0.1166	−0.0306
	9×9	1.7096	0.4060	0.5765	0.1635	0.1165	−0.0308
	11×11	1.7095	0.4059	0.5764	0.1635	0.1165	−0.0308
	13×13	1.7095	0.4059	0.5764	0.1635	0.1165	−0.0308
	Reddy and Chao‡	1.7100	0.4059	0.5765	0.1963	0.1398	−0.0308
	Pandya and Kant§	1.7054	0.4121	0.5829	0.1590	0.1132	−0.0308
	Pagano and Hatfield¶	1.9368	0.7200	0.6630	0.2920	0.2190	−0.0467
0.10	5×5	0.7166	0.5530	0.3859	0.0980	0.1325	−0.0273
	7×7	0.6600	0.4960	0.3601	0.1080	0.1391	−0.0240
	9×9	0.6628	0.4990	0.3615	0.1076	0.1388	−0.0241
	11×11	0.6627	0.4989	0.3614	0.1077	0.1388	−0.0241
	13×13	0.6627	0.4989	0.3614	0.1077	0.1388	−0.0241
	Reddy and Chao	0.6628	0.4989	0.3615	0.1292	0.1667	−0.0241
	Pandya and Kant	0.6613	0.5063	0.3653	0.1047	0.1348	−0.0242
	Pagano and Hatfield	0.7370	0.5590	0.4010	0.1960	0.3010	−0.0275
0.05	5×5	0.5467	0.5814	0.3215	0.0824	0.1381	−0.0256
	7×7	0.4884	0.5245	0.2942	0.0908	0.1460	−0.0219
	9×9	0.4913	0.5274	0.2957	0.0906	0.1457	−0.0221
	11×11	0.4912	0.5273	0.2957	0.0906	0.1457	−0.0221
	13×13	0.4912	0.5273	0.2957	0.0906	0.1457	−0.0221
	Reddy and Chao	0.4912	0.5273	0.2957	0.1087	0.1749	−0.0221
	Pandya and Kant	—	—	—	—	—	—
	Pagano and Hatfield	0.517	0.543	0.308	0.156	0.328	−0.0230
0.01	5×5	0.4901	0.5921	0.2973	0.0765	0.1402	−0.0250
	7×7	0.4309	0.5354	0.2690	0.0842	0.1486	−0.0211
	9×9	0.4338	0.5383	0.2705	0.0840	0.1483	−0.0213
	11×11	0.4337	0.5382	0.2705	0.0840	0.1483	−0.0213
	13×13	0.4337	0.5382	0.2705	0.0840	0.1483	−0.0213
	Reddy and Chao	0.4337	0.5382	0.2705	0.1009	0.1780	−0.0213
	Pandya and Kant	0.4322	0.5416	0.2704	0.0836	0.1444	−0.0214
	Pagano and Hatfield	0.4347	0.5390	0.2710	0.1390	0.3390	−0.0214

† $m_1 = \dfrac{100h^3E_2}{p_0a^4}$; $m_2 = \dfrac{h^2}{p_0a^2}$; $m_3 = \dfrac{h}{p_0a}$.

‡ Solutions of the first-order shear deformation by Reddy and Chao (1981).

§ FEM solutions based on Mindlin theory by Pandya and Kant (1988). The locations for the deflections and the stresses are: $w(0.5a, 0.5a)$, $\sigma_{xx}(0.4718a, 0.4718a, 0.5h)$, $\sigma_{yy}(0.4718a, 0.4718a, 0.25h)$, $\tau_{yz}(0.4472a, 0.0528a, 0.0)$, $\tau_{xz}(0.0528a, 0.4472a, 0.0)$, $\tau_{xy}(0.0282a, 0.0282a, 0.5h)$.

¶ Three-dimensional elasticity solutions by Pagano and Hatfield (1972).

equations using the finite element method. It is found that for all the transverse shear stresses τ_{xz} and τ_{yz}, the discrepancies between the present results and those by Pandya and Kant (1988) are within 3%. It should be noticed that Pandya and Kant (1988) presented their results in the Gauss points which are very close to, but not exactly the same as, the points we employed here. Similarly good agreement between the present results and those by three-dimensional elasticity theory (Pagano and Hatfield, 1972) is obtained.

Tables 2 and 3 present some numerical results of central deflections and axial centre stresses for a three-layer cross-ply (0°/90°/0°) laminate subject to a uniformly distributed load. These results are used to demonstrate the effects of thickness-to-span ratio h/a, length-to-width ratio a/b, and various edge restraint conditions on the convergent rate of the DQ method. It has been observed that neither the length-to-width ratio nor the thickness-to-span ratio has significant influences on the convergent rate. The rate of convergence is also quite independent of the boundary conditions. However, as the plate is subjected to a uniformly distributed load, the convergent rate of the DQ method is much slower than that where a sinusoidal load is applied. For the later case, the results obtained using 17 × 17 grid points are the convergent values, however, 13 × 13 grid points can produce results within the acceptable accuracy.

Bending behaviours of three-layer cross-ply (0°/90°/0°) laminates with arbitrary combination of boundary conditions are investigated using the DQ method, with the focus on

Table 2. Convergent rate of centre deflections $\bar{w}$† of three-layer cross-ply (0°/90°/0°) laminates with different edge conditions under uniform pressure ($h_i = h/3$, i = 1, 2 and 3; Material II)

a/b	h/a	Grid points	SSSS	SCSC	SFSF	SSSC	SSSF	SCSF
3	0.2	5×5	6.3860	3.8780	219.53	4.9436	44.012	17.155
		7×7	6.5078	3.8926	219.69	4.9954	47.810	18.606
		9×9	6.5201	3.9068	219.59	5.0084	48.256	18.750
		11×11	6.5244	3.9108	219.53	5.0125	48.298	18.759
		13×13	6.5256	3.9114	219.51	5.0134	48.305	18.761
		15×15	6.5259	3.9115	219.50	5.0137	48.305	18.761
		17×17	6.5260	3.9116	219.50	5.0137	48.305	18.760
3	0.14	5×5	13.880	6.7048	589.91	9.3268	103.57	35.717
		7×7	14.323	6.7258	590.13	9.4719	119.05	40.747
		9×9	14.379	6.7497	590.05	9.5051	121.55	41.271
		11×11	14.394	6.7565	589.96	9.5148	121.80	41.308
		13×13	14.398	6.7575	589.92	9.5165	121.82	41.310
		15×15	14.399	6.7576	589.90	9.5169	121.82	41.310
		17×17	14.399	6.7577	589.89	9.5171	121.82	41.310
5	0.2	5×5	1.6128	1.2257	222.30	1.4421	23.441	4.9338
		7×7	1.6103	1.2189	222.37	1.4376	24.258	5.0388
		9×9	1.6112	1.2199	222.33	1.4385	24.385	5.0711
		11×11	1.6112	1.2201	222.28	1.4386	24.412	5.0756
		13×13	1.6113	1.2202	222.24	1.4386	24.418	5.0761
		15×15	1.6113	1.2202	222.22	1.4386	24.420	5.0763
		17×17	1.6113	1.2202	222.22	1.4387	24.421	5.0764
5	0.14	5×5	3.0301	1.9036	598.57	2.4068	53.370	9.3863
		7×7	3.0298	1.8894	598.86	2.3969	57.468	9.6746
		9×9	3.0319	1.8909	598.72	2.3988	58.048	9.7483
		11×11	3.0323	1.8911	598.57	2.3991	58.137	9.7583
		13×13	3.0325	1.8913	598.48	2.3993	58.154	9.7593
		15×15	3.0326	1.8913	598.44	2.3993	58.160	9.7596
		17×17	3.0326	1.8913	598.42	2.3993	58.162	9.7597

†$\bar{w} = 10^6 \times (w/p_0)$, a = 200 in.

Table 3. Convergent rate of axial centre stresses $\bar{\sigma}_{xx}$† of three-layer cross-ply (0°/90°/0°) laminates with different edge conditions under uniform pressure ($h_i = h/3$, i = 1, 2 and 3; Material II)

a/b	h/a	Grid points	SSSS	SCSC	SFSF	SSSC	SSSF	SCSF
3	0.2	5×5	0.9344	0.4059	19.258	0.6304	3.7162	0.9644
		7×7	0.9264	0.3725	19.266	0.6063	3.9827	0.9654
		9×9	0.9332	0.3799	19.254	0.6133	4.0165	0.9701
		11×11	0.9344	0.3811	19.248	0.6146	4.0195	0.9703
		13×13	0.9348	0.3813	19.247	0.6149	4.0200	0.9704
		15×15	0.9349	0.3814	19.246	0.6150	4.0201	0.9704
		17×17	0.9350	0.3814	19.246	0.6150	4.0201	0.9704
3	0.14	5×5	1.9895	0.7869	39.145	1.2238	7.3834	1.6519
		7×7	1.9480	0.6937	39.139	1.1417	8.1452	1.5160
		9×9	1.9574	0.6969	39.116	1.1470	8.2758	1.5057
		11×11	1.9578	0.6955	39.109	1.1463	8.2885	1.5045
		13×13	1.9580	0.6953	39.107	1.1463	8.2897	1.5045
		15×15	1.9580	0.6953	39.106	1.1462	8.2898	1.5045
		17×17	1.9580	0.6953	39.105	1.1462	8.2898	1.5045
5	0.2	5×5	0.3231	0.1241	19.216	0.2353	1.3974	−0.0121
		7×7	0.3176	0.1099	19.216	0.2258	1.4088	−0.0752
		9×9	0.3192	0.1125	19.211	0.2277	1.4178	−0.0741
		11×11	0.3190	0.1125	19.207	0.2276	1.4193	−0.0743
		13×13	0.3191	0.1128	19.204	0.2278	1.4197	−0.0743
		15×15	0.3192	0.1128	19.202	0.2279	1.4198	−0.0743
		17×17	0.3192	0.1128	19.202	0.2279	1.4198	−0.0743
5	0.14	5×5	0.6709	0.2464	39.167	0.4360	3.1692	−0.0704
		7×7	0.6512	0.2183	39.171	0.4110	3.2466	−0.2513
		9×9	0.6553	0.2228	39.160	0.4153	3.2691	−0.2671
		11×11	0.6548	0.2220	39.152	0.4147	3.2713	−0.2703
		13×13	0.6549	0.2221	39.146	0.4148	3.2718	−0.2706
		15×15	0.6549	0.2221	39.144	0.4148	3.2719	−0.2707
		17×17	0.6549	0.2221	39.143	0.4148	3.2720	−0.2707

† $\bar{\sigma}_{xx} = \sigma_{xx}/p_0$, a = 200 in.

Table 4. Centre deflections $\bar{w}$† of three-layer cross-ply (0°/90°/0°) laminates with different edge conditions using 13 × 13 grid points ($h_i = h/3$, i = 1, 2 and 3; Material II)

a/b	h/a	Load	CCCC	CCCS	CCCF	CFCF	CFCS	CFFF	CFSF
1	0.2	UN	41.759	48.237	53.846	66.385	61.080	719.19	106.02
		SS	29.474	33.196	35.432	41.747	39.598	297.66	61.101
1	0.14	UN	92.728	107.24	118.48	144.89	134.12	1943.0	252.34
		SS	65.972	74.281	78.811	92.160	87.796	795.27	145.25
3	0.2	UN	3.8696	4.9668	15.665	68.715	34.050	736.57	111.84
		SS	2.9418	3.6406	8.9498	36.571	18.561	294.45	57.662
3	0.14	UN	6.6931	9.3867	17.638	150.94	73.894	2014.5	267.81
		SS	5.1227	6.8568	18.547	80.753	40.402	798.81	137.92
5	0.2	UN	1.2192	1.4383	4.9068	69.795	21.976	746.97	113.62
		SS	0.9639	1.1136	2.9794	36.923	11.689	298.29	58.450
5	0.14	UN	1.8902	2.3980	9.3808	154.14	46.964	2066.1	273.25
		SS	1.5026	1.8540	5.5568	82.046	25.008	818.85	140.56

†$\bar{w} = 10^6 \times (w/p_0)$, a = 200 in.

Table 5. Axial center stresses $\bar{\sigma}_{xx}$† of three-layer cross-ply (0°/90°/0°) laminates with different edge conditions using 13 × 13 grid points ($h_i = h/3$, i = 1, 2 and 3; Material II)

a/b	h/a	Load	CCCC	CCCS	CCCF	CFCF	CFCS	CFFF	CFSF
1	0.2	UN	4.1435	4.7793	5.1528	6.2252	5.8650	−18.996	9.8440
		SS	3.2521	3.6191	3.7829	4.3566	4.1948	−4.9343	6.1398
1	0.14	UN	8.6702	9.9436	10.641	12.692	12.018	−38.818	19.522
		SS	6.7949	7.5278	7.8312	8.9230	8.6261	−10.024	12.306
3	0.2	UN	0.3650	0.5975	0.8559	6.3667	2.9075	−19.356	10.050
		SS	0.3828	0.5296	0.7432	3.7026	1.8210	−5.6418	5.5042
3	0.14	UN	0.7059	1.1502	1.7638	12.839	6.0504	−40.316	19.832
		SS	0.7322	1.0323	1.4664	7.4739	3.7847	−11.864	10.895
5	0.2	UN	0.1108	0.2269	−0.0663	6.3713	1.2797	−19.556	10.060
		SS	0.1220	0.1960	0.1228	3.6352	0.8228	−5.7682	5.4474
5	0.14	UN	0.2219	0.4144	−0.1523	12.952	2.9760	−41.277	19.973
		SS	0.2364	0.3687	0.2062	7.3886	1.8742	−12.336	10.838

†$\bar{\sigma}_{xx} = \sigma_{xx}/p_0$, a = 200 in.

the cases where the analytical solutions are not available till now. Based on the above convergence studies, 13 × 13 grid points are used in the following computations.

In Table 4, the centre deflections for the three-layer cross-ply laminates are given with various boundary conditions. The amplitude of the centre deflection for the laminates greatly depends on the length-to-width ratio a/b and the thickness-to-span ratio h/a. The deflection increases with the boundary conditions varied from clamped, to simply supported, and finally to free. It is noted that with the ratio h/a kept constant (0.2 or 0.14), as the ratio a/b increases from 1 to 5, the centre deflection of the laminated plates with two parallel edges free increases monotonously when the plates are subjected to the uniformly distributed load; but does not change monotonously in the case when the plates are subjected to the sinusoidal distributed load. On the other hand, as for plates without the two parallel edges free, the central deflection decreases monotonously under either the sinusoidal load or uniform load. Table 5 displays the axial centre stresses of the three-layer cross-ply laminated plates with various boundary conditions. Similar conclusions to those given to the centre deflection can be drawn for the centre stresses. The only exception is that, for the laminated plates with two parallel edges free, with the ratio h/a kept unchanged (0.2 or 0.14), when increasing the ratio a/b from 1 to 5, the centre stress increases monotonously in the case of a uniformly distributed load; but decreases for CFSF and CFCF plates and increases for CFFF plates, monotonously, in the case of a sinusoidal distributed load.

5. CONCLUDING REMARKS

The differential quadrature technique has been employed for the static analysis of laminated composite plates subjected to transverse loads. The plates considered are of moderately thickness and with symmetric cross-ply laminates. The first-order shear deformation theory is used in the study with the governing differential equations transformed into a set of linear algebraic equations by the differential quadrature formulation. By implementing the boundary conditions in the discrete edge grid points, the resulting equations are solved and the solutions of the problems are obtained. The static responses are evaluated for several examples and comparison studies are carried out to assess the capability of the differential quadrature formulation in thick laminated plates modelling. It is concluded that the DQ method provides accurate solutions with a reasonable number of grid points. Thus the DQ method can serve as a powerful alternative to the numerical modelling of composite laminates.

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